SUGGESTED SOLUTIONS TO HOMEWORK I

Solution 1. (1) The solution is

$$u(t,x) = \sin x \cos t + t(x+1) + (1-\cos t)\cos x.$$

(2) The solution is

$$u(t,x) = x^2 + t^2 + t\sin y.$$

(3) The solution is

$$u(t,x) = x^2 + y^2 - 2z^2 + t + t^2 xyz.$$

Solution 2. (1) The solution is

$$u(t,x) = \cos\frac{3\pi t}{l}\sin\frac{3\pi x}{l} + \sum_{k=1}^{\infty}\frac{4l^3(1+(-1)^{k+1})}{k^4\pi^4}\sin\frac{k\pi t}{l}\sin\frac{k\pi x}{l}.$$

(2) The solution is

$$u(t,x) = \frac{8h}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cos\frac{(2k+1)\pi t}{2l}\sin\frac{(2k+1)\pi x}{2l}.$$

Solution 3. (1) Fix a point $(t_0, x_0) \in T$, integrate the inequality over $T_0 := \{(t,x) : x_0 - t_0 < x - t, x + t < x_0 + t_0, t > 0\}$, then

$$\int_{T_0} \partial_t^2 u - \partial_x^2 u dx dt \leq 0.$$

By Green's formula,

$$\begin{aligned}\int_{T_0} \partial_t^2 u - \partial_x^2 u dx dt &= \int_{\partial T_0} -\partial_t u dx - \partial_x u dt \\ &= -\int_{x_0-t_0}^{x_0+t_0} \partial_t u(0,x)dx + 2u(t_0,x_0) - u(0,x_0+t_0) - u(0,x_0-t_0),\end{aligned}$$

therefore

$$u(t_0,x_0) = \frac{u(0,x_0+t_0)+u(0,x_0-t_0)}{2} + \int_{x_0-t_0}^{x_0+t_0}\partial_t u(0,x)dx + \int_{T_0}\partial_t^2 u - \partial_x^2 u dx dt.$$

Since $\partial_t u(0,x) \leq 0$ and $\partial_t^2 u - \partial_x^2 u \leq 0$, we have

$$u(t_0,x_0) \leq \max_{0\leq x\leq 1} u.$$

(2) The solution is

$$u(t,x) = \frac{1}{\pi}\sin\pi x \sin\pi t.$$

Then

$$\max_{\overline{T}} u = u(\frac{1}{2},\frac{1}{2}) = \frac{1}{\pi},$$

however,

$$\max_{0\leq x\leq 1} u(0,x) = 0.$$

Solution 4. (1) By direct computation,

$$\begin{aligned}\frac{dE(t)}{dt} &= \int_{\mathbb{R}^n} \partial_t u \cdot \partial_t^2 u + \nabla u \cdot \partial_t \nabla u + f(u)\partial_t u dx \\ &= \int_{\mathbb{R}^n} \partial_t u(\partial_t^2 u - \Delta u + f(u))dx \\ &=0,\end{aligned}$$

which implies $E(t)$ is constant for all $t \geq 0$.

(2) By direct computation,

$$\begin{aligned}\frac{de(t)}{dt} &= \int_{B(x_0,t_0-t)} \partial_t u \cdot \partial_t^2 u + \nabla u \cdot \partial_t \nabla u + f(u)\partial_t u dx \\ &\quad - \int_{\partial B(x_0,t_0-t)} \frac{1}{2}(|\partial_t u|^2 + |\nabla u|^2) + F(u) dS_x \\ &= \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial n}\partial_t u - \frac{1}{2}(|\partial_t u|^2 + |\nabla u|^2) - F(u) dS_x \\ &= - \int_{\partial B(x_0,t_0-t)} \frac{1}{2}|n\partial_t u - \nabla u|^2 + F(u) dS_x\end{aligned}$$

(3) Since $F \geq 0$, by (2), we have

$$\frac{de(t)}{dt} \leq 0,$$

which implies that $e(t)$ is nonincreasing. Moreover, since $u(0,x) = \partial_t u(0,x) \equiv 0$ for all $x \in B(x_0,t_0)$, we have $e(0) = 0$. Hence $e(t) \equiv 0$ for all $t \geq 0$, then $\partial_t u = \nabla u \equiv 0$ for all $\{(t,x) : 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}$ and therefore $u \equiv 0$ for all $\{(t,x) : 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}$.